The optimal entropy bound and the self-energy of test objects in the vicinity of a black hole.

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Recently Bekenstein and Mayo conjectured an entropy bound for charged rotating objects. On the basis of the No-Hair principle for black holes, they speculate that this bound cannot be improved generically based on knowledge of other "quantum numbers", e.g. baryon number, which may be borne by the object. Here we take a first step in the proof of this conjecture. The proof make use of a gedanken experiment in which a massive object endowed with a scalar charge is lowered adiabatically towards a Schwarzschild's black hole and than dropped into the black hole from some proper distance above the horizon. Central to the proof is the intriguing fact that the self-energy of the particle receives no contribution from the scalar charge. Thus the energy with which the object is assimilated consists of its gravitational energy alone. This of course agrees with the No-scalar-Hair principle for black holes: after the object is assimilated into the black hole, any knowledge of the scalar field properties is lost. Using the GSL, we reach the conclusion that the original entropy bound was not improved by the knowledge of the scalar charge. At the end we speculate on whether or not massive vector fields may serve in the tightening of the entropy bound.

I. INTRODUCTION

A number of years ago Bekenstein proposed a universal bound on the entropy of a macroscopic object of characteristic size R and energy E. This bound takes the form [1]:

$$S \leq 2\pi ER/\hbar \tag{1.1}$$

(here and henceforth we use units with $G = c = 1$). Even though the bound was originally inferred in the framework of black holes physics, there is independent support for its validity. It is readily satisfied for composites of nonrelativistic particles on account of the fact that the entropy of the assemblage is never far removed from the number of particles involved. Likewise, the validity of the bound was verified directly both numerically and analytically for free massless quantum fields encompassed in cavities of various shapes and topologies (see review by Bekenstein and Schiffer [2]). Moreover, the entropy bound was recovered by Zaslavskii from the properties of the acceleration radiation [3]. With respect to self-gravitating systems, Sorkin, Wald and Jiu [4] afford a convincing indication that the entropy bound (1.1) holds for thermal radiation on the threshold of gravitational collapse, while Zaslavskii [5] proves the bound for a system consisting of a static black hole in equilibrium with thermal radiation in a box.

For an object with spin s, electric charge e, maximal characteristics length scale R and proper energy E, Bekenstein and Mayo conjectured an improved entropy bound [6]

$$S \leq 2\pi \frac{\sqrt{E^2R^2 - s^2} - e^2/2}{\hbar}, \tag{1.2}$$

a synthesis of Zaslavskii's [7] and Hod's [8] bounds.

By virtue of the duality of electromagnetism, this optimal bound can be generalized to include magnetic monopole charge g; one should merely replace $e^2 \to e^2 + g^2$. In addition, a deeper question can be set forth about the prospect of giving generic bounds on entropy which are tighter than the conjectured bound (1.2) on account of the object having some conserved "quantum numbers" apart from e, g or s. The case in point would be a tighter bound for an object with definite and known baryon number. The aim of this work is to put forward evidence in support of the conjecture [6] that bound (1.2), with the extension to magnetic monopole, cannot be bettered generically. By "generically" we

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mean without knowledge of details about the object's structure and dynamics. When these are known, it is possible to compute, by means of statistical mechanics, bounds on the entropy which can be small compared to bound (1.1) -for example see [9]. But if no such information is used, we must go to the black hole derivation of the entropy bounds. The conjecture that bound (1.2) cannot be bettered is made specifically for this situation.

The structure of the paper is as follows. In Sec. II we briefly review the subject of entropy bounds as it manifest itself in the arena of black hole physics. In addition, we put forward the motivation for investigating the issue of entropy bounds with regard to scalar and vector fields. In Sec. III we begin by writing down the action functional for a massive particle coupled to a massless scalar field. The interaction action is chosen for simplicity and conformal invariance. The trajectories of test particles on the black hole spacetime background are best characterized by their constants of the motion. In Sec. IV we compute the energy of the particle as the constant of motion associated with the timelike Killing vector. We thus discuss the contribution of the scalar field to the energy of the test object by working out in closed form the scalar potential generated by a stationary point scalar charge in the background of a spherical static black hole. We use a simple procedure to regulate the potential. We find that after the regulation the scalar self-energy vanishes. Hence the scalar interaction contributes nought to the energetics of the process. This signifies that the object is assimilated with its gravitational energy only, *e.g.* if the process is an adiabatic one, then at the end we are left with a new spherical static black hole, whose mass is equal to the mass of the initial black hole augmented only by the gravitational energy of the object. The increase of the black hole mass obviously increases its horizon's surface area. In Sec. V we analyze the corrections to the area formula and find that the corrections must vanish in linear theory. The change in horizon area that results from the lowering of the object onto the horizon furnishes a derivation from the GSL of the entropy bound (1.1). We conclude by illustrating the application of the method used here to another problem: an entropy bound for a particle coupled to a massive vector field with a vanishingly small but non zero mass.

II. ENTROPY BOUNDS IN THE FRAMEWORK OF BLACK HOLE PHYSICS: THE STATE OF THE ART

In its original form, the entropy bound (1.1) is saturated by the Schwarzschild black hole. This prompted the observation [1] that the Schwarzschild black hole is the most entropic object for given size and energy. But the Kerr black hole's entropy falls below bound (1.1) (this will be true for any reasonable interpretation of the radius R of the nonspherical Kerr hole).

This asymmetric state of affairs motivated Hod [8] to search for a tighter bound on entropy for objects with angular momentum which is saturated by the Kerr hole. Hod repeats Bekenstein's derivation [10,11] of the minimal increment in Kerr-Newman (KN) horizon area that is caused by an object's infall. That derivation applied the idea of Christodoulou [12] together with Carter's [13,14] integrals of the Lorentz equations of motion to a particle of rest mass μ and radius R moving in a KN background. The minimal growth in horizon area was deduced from the conservation laws and the relation they establish between the change in black hole parameters and the energy and orbital angular momentum of a particle in an orbit, such that the particle's center of mass can get to distance R from the horizon. Remarkably, it turns out that the minimal area growth is independent of the black hole parameters. Because μ can be identified with the total proper energy of the object, bound (1.1) follows from the minimal area growth and the GSL.

The particle's spin was not taken into account in Carter's integrals. Hod refers instead to Hojman and Hojman's [15] integrals of motion for a neutral object with spin s moving on a KN background. Repeating the argument that led to the original entropy bound (1.1) and appealing to the GSL allows Hod to infer the entropy bound (1.2) with $e = 0$.

Recently many researches closed in on the derivation of Zaslavskii's [7] proposed bound for charged objects. Those derivations focused on absorption of the relevant object by a black hole, and on the concomitant change in horizon area. Hod [16] makes use of the thermodynamics of a Schwarzschild black hole, while Bekenstein and Mayo [6] makes use of the thermodynamics of a Reissner–Nordström (RN) black hole. Linet [17] utilizes the thermodynamics of a KN black hole, with similar results. At the center of the derivation lies the fact that a charged particle in a black hole's vicinity is affected by not only the Lorentz force from the black hole's electromagnetic field and by the Abraham–Lorentz–Dirac radiation reaction force, but also by the force originating from the black hole's polarization by the particle's electric field. Now it is known that a particle at rest in a static black hole background does not radiate (despite its being accelerated). For that reason, one can expect the radiation reaction force to vanish. This suggests that we should focus on an object lowered slowly from a large distance to the horizon. Under this circumstances, it is possible to suppose that only the gravitational, Coulomb and polarization forces act upon it. This approach allows the authors mentioned above to derive Zaslavskii's bound by use of the GSL. Now, *if*, as it is sometimes claimed,

the GSL functions independently of entropy bounds, there should not have been reason for this unusual effect (black hole polarization) to supplies precisely the missing element in the derivation of the entropy bound for charged objects from the GSL. This is yet another demonstration that the GSL provides a valid road to entropy bounds.

A useful by-product of the mentioned chain of generalizations is the revelation that the entropy bound is independent of the type of black hole employed in the calculations. Thus, it seems that one may choose the Schwarzschild spacetime as the simplest settings for the study of entropy bounds.

Bound (1.2) has one more merit; any KN black hole (mass m, charge q and angular momentum j) saturates it. The horizon area of such a black hole is [14]

$$A = 4\pi(r_+^2 + j^2/m^2); \qquad r_+ \equiv m + (m^2 - j^2/m^2 - q^2)^{1/2} \tag{2.1}$$

Substituting r_+, squaring as required, and cancelling terms gives

$$A = 2\pi(4mr_+ - q^2) = 2\pi[4m((r_+^2 + j^2/m^2) - j^2/m^2)^{1/2} - q^2] \tag{2.2}$$

In light of Eq.(2.1) it is reasonable to interpret $(r_+^2 + j^2/m^2)^{1/2}$ as the radius R of the hole. Incorporating this in the last equation and dividing by $4\hbar$ gives for the black hole entropy

$$S_{\rm BH} = \frac{2\pi}{\hbar}[(m^2R^2 - j^2)^{1/2} - q^2/2] \tag{2.3}$$

If we identify $m \leftrightarrow E$, $q \leftrightarrow e$ and $j \leftrightarrow s$, this is exactly the upper bound of Eq.(1.2). Hence the KN black hole saturates the proposed entropy bound. This property would be lost if the bound were modified. Hence we adopt it in the given form.

Finally, we arrive at the principal issue we hope to elucidate in this work.

Question: *Is it possible to improve bound (1.2) generically based on the knowledge of other "quantum numbers" which may be borne by the object?*

The No-Hair conjecture is central to our reasoning. A large amount of work has certified that a stationary black hole can have just a few parameters. The irrefutable ones are mass, charge, magnetic monopole and angular momentum. Skyrmion number is a possible addition [18], but one whose physical significance is unclear [11]. Other candidates [19–21] are associated with unstable black holes [22]. The technique we propose to use is perturbative in nature. So it stand to reason to focus only on black holes which remain stable despite the outside perturbations. On that account, we concentrate on the KN black holes with parameters m, q, g and j.

In [6], Bekenstein and Mayo give arguments against the prospect of obtaining generic bounds on entropy, which are tighter than bound (1.2) on account of the object having an extra additive conserved global quantity such as baryon or lepton numbers. Also excluded are short range fields, such as the short range, W-boson mediated, weak force. The third and last case considered is when the extra additive quantity carried by the object, b is the source of a long–range field schematically denoted by $\mathcal{B}$. For example, $\mathcal{B}$ can be a scalar field with presumably small or zero mass, or a massive vector field with vanishingly small but nonzero mass. The range may be finite but large in comparison to the size of a typical object. For a massive scalar or vector field this means, that on the one hand the Compton wavelength of the object itself must be very small on the scale of the hole, and on the other the scale of the hole must be small compared with the range of the field. This is granted by the smallness of the field's mass. Now the area formula may differ from the usual area formula for a KN black hole by terms depending on b, because of the perturbation that $\mathcal{B}$'s energy-momentum tensor exerts on the metric. Unless $\mathcal{B}$ is a gauge field which remains massless in the classical (or low energy) limit, we cannot rule out such dependence. This is because Birkhoff–type theorems exist only for massless vector fields, and from our point of view, the electromagnetic field is the only one. Thus, while the area stays constant during the descent as required by the adiabatic theorem, m may change by a quantity of $O(b^2)$. The sign of this quantity is vague without a specific model. Likewise, the energy with which the object is assimilated into the black hole, will presumably have a term of $O(b^2)$. Indeed, as before this term may be positive here. But it does not follow that the effect of b is to subdue the growth of the horizon area. This is because the indefinite sign of the correction to the area formula. Without calculating linear corrections to the metric, one cannot judge whether the change in area is incremented or depressed by b's presence.

In the following sections these questions will be precisely phrased and answered.

III. ACTION OF A MASSIVE PARTICLE COUPLED TO A CONFORMAL SCALAR FIELD.

We use the signature $\{-,+,+,+\}$ and denote the timelike coordinate outside the black hole, presumed to be a spherical static one, by x^0. The simplest parameter independent action functional for a particle of rest mass μ coupled to a conformal massless scalar field Φ is

$$S = -\int (\mu + b\Phi)\sqrt{-g_{\alpha\beta}\frac{dx^\alpha}{d\lambda}\frac{dx^\beta}{d\lambda}}d\lambda \tag{3.1}$$

where b, a constant, is the coupling strength, λ is a parameter, and $x^\alpha(\lambda)$ is the trajectory of the particle. The term proportional to μ is the action for a free particle; that proportional to b is the interaction action. The interaction chosen here is the most natural one in that the source term it generates for the wave equation for Φ is independent of Φ itself. Furthermore, the interaction action is invariant under the conformal transformation $g_{\mu\nu} \to \Omega^2 g_{\mu\nu}$, $\Phi \to \Phi\Omega^{-1}$, where Ω is an arbitrary function. Since the free field action possesses the same invariance [23] it follows that the wave equation with a source will be conformally invariant. Thus, the coupling envisaged here, apart from being the simplest one, is singled out by its invariance properties. The same cannot be said about the once popular derivative coupling ($\propto \Phi_{,\alpha}\, dx^\alpha/d\lambda$) [24] which is known to be unphysical on grounds of lack of renormalizability.

In a sense, the coupling as chosen in (3.1) is the analog of minimal coupling in electromagnetism. However, it should be noted that the analogy with the electromagnetic case stops here; conservation of charge is not obligatory for consistency of the Klein–Gordon equation, so there is no obstacle to permitting b to vary. However, here we suppose throughout that b is constant.

Variation of S with respect to x^ν gives the equation of motion

$$(\mu + b\Phi)\frac{D^2x^\nu}{d\lambda^2} = \left\{\frac{1}{2}(\mu + b\Phi)\frac{d}{d\lambda}\ln\Xi - b\Phi_{,\alpha}\frac{dx^\alpha}{d\lambda}\right\}\frac{dx^\nu}{d\lambda} - b\Xi\,\Phi^{,\nu} \tag{3.2}$$

where $\Xi = -g_{\alpha\beta}\frac{dx^\alpha}{d\lambda}\frac{dx^\beta}{d\lambda}$. Since the action S is invariant under a change of the parameter λ, we are at liberty to impose a condition on Ξ to fix the choice of λ. Two choices are of interest here. If we set $\Xi = 1$, λ becomes proper time τ, and Eq. (3.2) takes the form

$$(\mu + b\Phi)\frac{D^2x^\nu}{d\tau^2} = -b\left\{\Phi^{,\nu} + \Phi_{,\alpha}\frac{dx^\alpha}{d\tau}\frac{dx^\nu}{d\tau}\right\}. \tag{3.3}$$

A second useful choice is $\Xi = (1 + b\Phi/\mu)^2$ which makes the coefficient of $\frac{dx^\nu}{d\lambda}$ in Eq. (3.2) to vanish. In this case λ is just an affine parameter. The equation of motion is

$$\frac{D^2x^\nu}{d\lambda^2} = -\frac{b}{\mu}(1 + \frac{b}{\mu}\Phi)\Phi^{,\nu}. \tag{3.4}$$

In addition we have

$$-g_{\alpha\beta}\frac{dx^\alpha}{d\lambda}\frac{dx^\beta}{d\lambda} = \left(\frac{d\tau}{d\lambda}\right)^2 = (1 + \frac{b}{\mu}\Phi)^2 \tag{3.5}$$

which shows that in regions of weak Φ, λ is essentially proper time.

IV. THE ENERGETICS OF THE PROCESS IS UNAFFECTED BY THE SCALAR FIELD

Consider now a test scalar charge moving in the background of black hole spacetime which possesses a symmetry represented by a Killing vector ξ^α. So $\xi_{(\alpha;\beta)} = 0$ and $\xi^\alpha\Phi_{,\alpha} = 0$. The scalar-charge trajectory, that of a particle obeying the equation of motion Eq. (3.2), is best characterized by its constants of motion. It is easy to show from Eq. (3.3) and Eq. (3.4) that

$$\begin{aligned}\mathcal{E} &\equiv -\xi_\alpha\frac{dx^\alpha}{d\tau}(\mu + b\Phi)\\ E &\equiv -\xi_\alpha\frac{dx^\alpha}{d\lambda}\end{aligned} \tag{4.1}$$

are constants of the motion in the proper time parameterizations and in the affine parameter parameterizations respectively . Note that $\mathcal{E} = \mu E$. The stationarity of the envisaged background fixes the form of the timelike Killing vector to be $\xi^\alpha = \delta^\alpha_t$ for which $\mathcal{E}$ reduces to

$$\mathcal{E} = -(\mu + b\Phi)g_{0\beta}\frac{dx^\beta}{d\tau}. \tag{4.2}$$

This corresponds to the usual notion of energy as measured at infinity. Its first term expands to $\mu+\frac{1}{2}\mu(d\mathbf{x}/dt)^2$ in the Newtonian limit. The second term, $-b\Phi$, is thus the scalar potential energy.

In our gedanken experiment the object of rest mass μ and scalar charge b, idealized as spherically symmetric, is suspended by some means to keep it from falling freely, and is slowly lowered radially towards the black hole. Of course, the forces restraining its fall change its energy measured at infinity as it is lowered. The idea is to bring the object as close to the horizon as possible, and then drop it in, inferring from the energy measured at infinity at its last prefall position the increase in horizon area that this causes. A complication - the Unruh–Wald buoyancy in acceleration radiation [25,26] - may cause the object to float neutrally some distance from the horizon, thus arresting the contemplated descent. But as demonstrated by Bekenstein [27–29], provided the number of relevant particle species in nature is not large (which seems to be true in our universe), and provided the object is macroscopic and composed of parts that obey quantum mechanics, the buoyancy is negligible all the way to very near the horizon, and makes no practical difference to the energy budget of the process.

Correct to $O(b^2)$ the metric may be taken as Schwarzschild's. In isotropic coordinates it is

$$ds^2=-\left(\frac{1-\frac{m}{2\rho}}{1+\frac{m}{2\rho}}\right)^2(dx^0)^2+(1+\frac{m}{2\rho})^4\left[d\rho^2+\rho^2(d\theta^2+\sin^2\theta\,d\phi^2)\right] \tag{4.3}$$

We see that the horizon resides at $\rho=m/2$. Because the object is nearly stationary, its 4-velocity, which we normalize to -1, must have the form $dx^\alpha/d\tau\approx\{(-g_{00})^{-1/2},0,0,0\}$. Substitution in Eq.(4.2) from the metric gives for the energy, when the object's CM is at $\rho=a$ and $\theta=0$,

$$\mathcal{E}=\left(1-\frac{m}{2a}\right)\left(1+\frac{m}{2a}\right)^{-1}(\mu+b\Phi)_{\rho=a,\theta=0} \tag{4.4}$$

As elucidated by Vilenkin [30] and corroborated by Smith and Will [31], by contrast to the situation in flat spacetime, in the presence of a Schwarzschild black hole the self–energy of an electric charge measured at infinity is modified. As we intend to show, this is surprisingly *not* so for the scalar field.

Eq.(A7) of the Appendix gives $\Phi(\rho,\theta)$, the scalar potential due to a stationary (or nearly so) point scalar charge of strength b in the background of a spherical black hole. This expression, accurate to $O(b^2)$, is the scalar analog of an early brilliant solution by Copson [32] who found the electric field resulting from a charge in the Schwarzschild background. The solution (A7) was also found independently by Linet [33].

$\Phi(\rho,\theta)$ naturally diverges at $\rho=a$ and $\theta=0$, the position of the scalar charge. Thus if we want to use it for our finite object, we must regulate the potential before going to the limit $\rho\to a$ and $\theta\to 0$ as required by formula (4.4).

The simplest procedure is as follows (compare with [6]). We reexpress Φ in terms of new coordinates $\{\varrho,\vartheta,\phi\}$ centered on the charge, rather than on the black hole center, as was the case for $\{\rho,\theta,\phi\}$, but sharing the same polar axis. This implies the substitutions

$$\begin{aligned}&\rho\cos\theta\to a+\varrho\cos\vartheta,\\&\rho\to\sqrt{a^2+\varrho^2+2a\varrho\cos\vartheta}.\end{aligned} \tag{4.5}$$

A small metric sphere of proper radius R located at $\{\rho,\theta\}=\{a,0\}$ is the coordinate sphere $\varrho=(1+\frac{m}{2a})^{-2}R;\forall\vartheta$. Since ϱ is the coordinate distance from the charge it makes sense to expand Φ in a Laurent series in ϱ:

$$\Phi=-\frac{b}{\varrho(1+\frac{m}{2a})^2}-\frac{b}{a(1+\frac{m}{2a})^2}\,\frac{\cos\vartheta}{(2a/m)^2-1}+O(\varrho). \tag{4.6}$$

The divergent term in Eq.(4.6) corresponds to the scalar potential of the charge b in flat spacetime; there we expect $\Phi=-b\varrho^{-1}$. A factor $(1+\frac{m}{2a})^{-2}$ is required to convert the coordinate distance ϱ to proper distance. Thus when taking the limit $\rho\to a$ and $\theta\to 0$ ($\varrho\to 0$) of Φ, we must discard the first term in the r.h.s. of Eq.(4.6).

Our spherically symmetric finite object samples all directions about its center without discrimination. Because the metric also looks isotropic in coordinates $\{\varrho,\vartheta,\phi\}$, we must thus average out the second term in the r.h.s. of Eq.(4.6) over all angles ϑ and ϕ; as a result its contribution vanishes. Terms of $O(\varrho)$ vanish as the size of the object shrinks. Thus the entire scalar contribution to $\mathcal{E}$ vanishes! This result was also found independently by Zelnikov and Frolov [34], Wiseman [35] and Burko [36]. Substituting this in Eq.(4.4) we find

$$\mathcal{E}=\mu\left(1-\frac{m}{2a}\right)\left(1+\frac{m}{2a}\right)^{-1}+O\left(\frac{b^3}{m^2}\right) \tag{4.7}$$

where we have included the next higher order correction to the energy due to the coupling to the scalar field.

When the object is near the horizon, the proper distance from its CM to the horizon is

$$\ell \equiv \int_{m/2}^{a} (g_{\rho\rho})^{1/2} d\rho \approx 4(a - m/2) + O[(a - m/2)^2]. \tag{4.8}$$

Expressing a in Eq. (4.7) in terms of ℓ by means of Eq.(4.8) we get

$$\mathcal{E} = \left(\frac{\mu\ell}{4m}\right)\left[1 + O\left(\frac{\ell}{m}\right)\right] + O\left(\frac{b^3}{m^2}\right). \tag{4.9}$$

Corrections of $O(\ell/m)$ are duly neglected, as are those of $O(b^3/m^2)$ by virtue of the assumed smallness of ℓ and b compared to the large mass of the black hole. The gradual approach to the horizon must stop when the proper distance from the object's CM to the horizon reaches the object's radius R. Hence,

$$\mathcal{E} \geq \frac{\mu R}{4m}. \tag{4.10}$$

V. THE AREA FORMULA AND THE ENTROPY BOUND

As mentioned, our primary concern is with changes in the horizon area. The area formula must be corrected for the perturbation of the metric originating in the object, which in linear approximation should be of $O(\mu)$ and $O(b^2)$, the first caused by the energy momentum tensor of the object's mass, and the second by the energy momentum tensor of the scalar field. We now argue that the corrections to the area formula actually appear only in the next higher order.

First suppose the area A was in fact perturbed in linear approximation to $O(\mu)$ and $O(b^2)$. By spherical symmetry of the background these corrections would not depend on the direction along which the object was lowered. If N equal bodies were lowered, each along a different radial direction, the perturbation would be N times larger by linearity of the approximation. But if enough bodies were disposed on a spherical shell concentric with the black hole, the metric perturbation due to the energy momentum tensor of *the object's mass* at the horizon should tend to zero by Birkhoff's theorem [14] that the metric exterior to a spherical black hole is exactly Schwarzschild if the surroundings are spherically symmetric. We thus get a contradiction unless we admit that the perturbation of $O(\mu)$ vanish in linear theory. Any corrections to A must thus be of higher order, like $O(\mu^2)$, *etc.*

What about the *scalar* perturbation? Since there is no analog of Birkhoff's theorem for scalar fields, we must verify directly that the perturbation to the horizon area formula is of order higher than $O(b^2)$.

As before we assume that a large number of scalar charges are disposed on a spherical shell concentric with the black hole. By the linearity of the approximation the perturbation to the metric from the scalar charge's field should be of $O(b^2)$. Now, a spherically symmetric perturbation to a static spacetime surrounding a black hole can be expressed by

$$\begin{aligned} &g_{\mu\nu} \approx g^{(B)}_{\mu\nu} + h_{\mu\nu}, \\ &h_{\mu\nu}\, dx^\mu\, dx^\nu = -u(\rho)\left(dx^0\right)^2 + f(\rho)(d\rho^2 + \rho^2 d\Omega^2) \end{aligned} \tag{5.1}$$

where $g^{(B)}_{\mu\nu}$ is the background metric and $d\Omega^2$ is the background line element on the 2-sphere, $d\Omega^2 = d\theta^2 + \sin\,\theta^2 d\phi^2$.

The horizon area formula is

$$A = \int \sqrt{-g}_{\rho=\rho_{\mathcal{H}}}\, d\theta d\phi \tag{5.2}$$

where $\sqrt{-g}$ and $\rho_{\mathcal{H}}$ are to be evaluated at the (perturbed) location of the horizon. The correction to A in linear theory can be determined by the following procedure. To first approximation, the volume element is

$$g = det\,(g_{\mu\nu}) = \epsilon^{\alpha\beta\gamma\delta}\, g_{0\alpha} g_{\rho\beta} g_{\theta\gamma} g_{\phi\delta} = g^{(B)}(1 + h) + O(b^4) \tag{5.3}$$

where $\epsilon^{\alpha\beta\gamma\delta}$ is the Levi-Civita tensor and $h = g^{(B)}_{\alpha\beta}\, h^{\alpha\beta}$. Terms of $O(b^4)$ are to be understood as quadratic in h. Using Eq. (5.2) for the area formula, we find

$$A = A^{(B)}\left(1 + \frac{1}{2}\int h_{\rho=\rho_{\mathcal{H}}}\, d\theta d\phi\right) + O(b^4). \tag{5.4}$$

Where does the new horizon resides? The point $\rho = \rho_{\mathcal{H}}$ where g_{00} vanishes is to be interpreted as the location of the horizon (if several zeros exist the location of the horizon corresponds to the outermost one). Therefore, correct to $O(b^4)$ the new horizon resides at

$$\rho_{\mathcal{H}} = \rho_{\mathcal{H}}^{(B)} + \delta\rho = \rho_{\mathcal{H}}^{(B)} - \left.\frac{h_{00}}{(g_{00}^{(B)})'}\right|_{\rho=\rho_{\mathcal{H}}^{(B)}} + O(b^4). \tag{5.5}$$

Hence, correct to $O(b^2)$, h in Eq. (5.4) can be evaluated at $\rho = \rho_{\mathcal{H}}^{(B)}$.

Now, the field equation for $h_{\mu\nu}$ are as follows [14]. First the Ricci tensor due to the perturbation is

$$\begin{aligned} R_{\alpha\beta} &= R_{\alpha\beta}^{(B)} + R_{\alpha\beta}^{(1)}(h) + O(b^4), \\ R_{\alpha\beta}^{(1)}(h) &= \frac{1}{2}\left(-h_{;\alpha\beta} - h_{\alpha\beta;\gamma}{}^{\gamma} + h_{\gamma\alpha;\beta}{}^{\gamma} + h_{\gamma\beta;\alpha}{}^{\gamma}\right) \end{aligned} \tag{5.6}$$

where $R_{\alpha\beta}^{(B)}$ is the background Ricci tensor, which vanishes for the Schwarzschild spacetime. Next, a useful identity can be established

$$R = h^{\alpha\beta}{}_{;\alpha\beta} - h_{;}{}^{\beta}{}_{\beta} + O(b^4). \tag{5.7}$$

From the trace of Einstein's equations, $R = -8\pi T$, where $T = -\Phi_{,\alpha}\Phi^{,\alpha}$ is the trace of the energy momentum tensor of the massless spherically symmetric scalar field. It is an invariant of the geometry. By Eq. (A9) of the Appendix with charge distribution density appropriate for a spherical symmetric configuration of scalar charges, T is $O(b^2)$. The invariance of T and hence of R signifies that their values at the perturbed horizon (as given in Eq. (5.5)) which is by all means a physically regular surface, must both be finite. Else their divergence would give rise to curvature singularity at the horizon, a thing that would render our perturbation approach invalid. Taylor expanding $R^{(1)}$ around $\rho^{(B)}$ we find

$$R^{(1)}(\rho_{\mathcal{H}}^{(B)} + \delta\rho) = R^{(1)}(\rho_{\mathcal{H}}^{(B)}) + (R^{(1)}(\rho_{\mathcal{H}}^{(B)}))'\,\delta\rho + O(b^6). \tag{5.8}$$

The second term in the r.h.s. of the equation is obviously $O(b^4)$ (see Eq. (5.5)), hence it can not cancel any divergence due to the first term in the r.h.s. of the same equation, which is $O(b^2)$. To put it in other words, assuming that R is an analytic function of the coordinates, its expansion in powers of b must be bounded term by term. A straightforward calculation yields

$$\begin{aligned} R^{(1)} &= \mathcal{F}_0\, f''(\rho) + \frac{\mathcal{F}_1\, f(\rho) + \mathcal{F}_2\, f'(\rho)}{(1-m/2\rho)} \\ &\quad + \frac{\mathcal{U}_0\, u(\rho)}{(1-m/2\rho)^4} + \frac{\mathcal{U}_1\, u'(\rho)}{(1-m/2\rho)^3} + \frac{\mathcal{U}_2\, u''(\rho)}{(1-m/2\rho)^2} \end{aligned} \tag{5.9}$$

where $' = d/d\rho$ and $\mathcal{F}_i$ and $\mathcal{U}_j$ are known functions of ρ, finite at $\rho = \rho_{\mathcal{H}}^{(B)}$. Here $g_{\alpha\beta}^{(B)}$ is taken as in Eq. (4.3). An examination of the expression above confirms that for R to be finite on the horizon, $f(\rho)$ must vanish on the horizon at least as fast as $(1-m/2\rho)^2$ and $u(\rho)$ must vanish at least as fast as $(1-m/2\rho)^4$. What does this suggest for the correction to the horizon area formula, Eq. (5.4)?

Using the metric Eq. (4.3) we work out the expression for h with the subsequent simple result

$$h = \frac{(1+m/2\rho)^2}{(1-m/2\rho)^2}u(\rho) + \frac{3f(\rho)}{(1+m/2\rho)^4}. \tag{5.10}$$

Considering the fact that we are really interested in the value of h on the horizon at its background position, namely $\rho = \rho_{\mathcal{H}}^{(B)}$, we are faced with the observation that h *vanishes* on the horizon! Hence the area formula is left unperturbed in linear approximation: any corrections to A must be of higher order, like $O(b^4)$, *etc.* Hence by Eq.(2.1) (with $q = j = 0$)

$$A = 16\pi m^2 + O(b^4/m^2) + O(\mu^2) \tag{5.11}$$

where we have included all possible second order terms of the correct dimensions.

The descent of the object, if sufficient slow, is known to be an adiabatic process which causes no change in the horizon area [37]. It follows that to the stated accuracy, m is unchanged in the course of the lowering process because

A is preserved. When the object is finally dropped, and absorbed by the black hole, m increases by $\mathcal{E}$; after the suspension machinery has been adiabatically retrieved, we acquire an unperturbed Schwarzschild black hole with mass $m + \mathcal{E}$. Calculating its horizon area from Eq. (5.11) and subtracting the area of what was an unperturbed Schwarzschild black hole of mass m, we find the change

$$\Delta A = 32\pi m\mathcal{E} + O(\mathcal{E}^2) + \cdots . \tag{5.12}$$

Finally substitution of Eq.(4.10) gives

$$\Delta A \geq 8\pi\mu R \left[1 + O\left(\frac{\mu R}{m^2}\right)\right] + \cdots . \tag{5.13}$$

Notice that the black hole parameter m has dropped out from the dominant terms, in analogy with results for uncharged objects [10]. The minimum change in black hole entropy, $\Delta A/4\hbar$ with the equal sign, is thus a property of the object itself. The entropy of the object cannot exceed this amount, lest the overall entropy of the world decrease upon the object's assimilation (see [27] for the irrelevancy of buoyancy corrections in this connection). We thus find the bound on the entropy of an object of scalar charge b, characteristic size R and proper energy $E = \mu$ to coincide with Bekenstein's proposal Eq. (1.1). It wasn't improved by knowledge of the object's scalar charge.

Actually this result can be easily generalized to the case of an object with mass μ, endowed with a *scalar* charge b and an *electromagnetic* charge e, which is assimilated by a RN black hole with charge q. In order to include the electromagnetic effects one should merely add to the action functional (3.1) an interaction term of the form

$$S_{Int} = e \int \hat{A}_\alpha \, \dot{x}^\alpha d\tau \tag{5.14}$$

where, as in Sec. III, $x^\alpha(\tau)$ denotes the particle's trajectory, τ the proper time, an overdot stands for $d/d\tau$, and $\hat{A}_\alpha$ means the electromagnetic 4–potential with the self–field of the particle subtracted off and then evaluated at the particle's spacetime position. The energy is now

$$\mathcal{E} = -(\mu + b\Phi) g_{0\beta} \, \dot{x}^\beta - e \left(\hat{A}_0^{(q)} + \frac{1}{2} A_0^{(e)}\right) . \tag{5.15}$$

Here $\hat{A}_0^{(q)}$, linear in q, is produced by the black hole and $A_0^{(e)}$, whose source is the object itself, is linear in e. The factor $\frac{1}{2}$ takes care of the fact that the object owes part of its energy to its own field, not to the background one. As before, $\mathcal{E}$ corresponds to the usual notion of energy as measured at infinity.

We require that q, e and b be very small on the scale of m, the mass of the hole. Then, correct to $O(e)$ which we regard as the same as $O(b)$ and $O(q)$, the metric may be taken as Schwarzschild's. Retracing the steps of the derivation in the previous section we find that the energy of the object at a proper distance equal to the object's proper radius R is

$$\mathcal{E} \geq \frac{2\mu R + e^2 + 4eq}{8m}. \tag{5.16}$$

The scalar field parameter is again missing due the vanishing of the scalar self-energy.

Thus, we can declare that the entropy bound (1.2) is left intact, not least due to the fact that the Birkhoff's theorem applies in the case of the electromagnetic field. Coupling of the scalar field to the field generated by the electric charges, should not open a loophole in the above claim. This is because the corrections to the electromagnetic and scalar fields due to the scalar-electromagnetic interaction are of second order in the coupling constant. Therefore, if we take that to be of the same order of magnitude as $O(b)$, then these corrections would induce corrections to the area formula of $O(b^4)$, which are duly neglected. Moreover, adding a mass term to the free scalar field action, should leave the entropy bound unaltered, provided the Compton wavelength of the scalar field is large on the scale of the black hole. Accordingly, terms in the scalar field equation, proportional to the mass of the field, can be neglected.

VI. SPECULATION: SOURCES OF MASSIVE VECTOR FIELDS AND THE OPTIMAL ENTROPY BOUND

What about sources of massive vector fields? It turns out that most of the results that were obtained for the massless scalar and vector fields may be used in this case. Vilenkin [30] points out that if instead of the electromagnetic field,

the particle is coupled to a vector-meson field of vanishingly small, but nonzero, mass, then it can be shown that the self force has the same magnitude but opposite direction. This sharp difference between massive and massless vector fields is a result of different boundary conditions at the horizon surface. The basic idea is as follows.

If the mass of the vector field A_ν is not exactly zero, then Maxwell equations have to be replaced by Proca's:

$$F^{\alpha\beta}{}_{;\beta} - \mathsf{m}^2 A^\alpha = 4\pi j^\alpha. \tag{6.1}$$

Here and henceforth $F_{\alpha\beta} = A_{\beta;\alpha} - A_{\alpha;\beta}$. We assume that the mass of the vector field is very small, namely m^{-1}, the Compton wavelength of the massive vector field, is much larger than any characteristic distance in the problem. Therefore correct to $O(\mathsf{m}^2 m^2)$, the mass term in the equation above can be neglected. In solving, let b be the strength of a charge of the massive vector field A_ν at a distance a from the black hole, and let $m/2 \ll \mathsf{m}^{-1}$. What boundary conditions must be fulfilled for the consistency of the solution? We require that invariants associated with the energy-momentum tensor of the field A_ν

$$T^\nu_\sigma = \frac{1}{4\pi}\left(F_{\sigma\alpha}F^{\nu\alpha} - \frac{1}{4}\delta^\nu_\sigma F^{\alpha\beta}F_{\alpha\beta} + \mathsf{m}^2\left(A_\sigma A^\nu - \frac{1}{2}\delta^\nu_\sigma A_\alpha A^\alpha\right)\right), \tag{6.2}$$

be nonsingular at the horizon, any divergence in these would induce divergences in the invariants of the geometry via Einstein's equations. The case in point would be that T, the trace of the energy-momentum (6.2), which is proportional to the invariant $A_\alpha A^\alpha$, must be bounded everywhere, and the potential A_0 must vanish at least like $(\rho - m/2)$ as $\rho \to m/2$. Thus all the physically meaningful solutions of the Proca field equation (6.1) must satisfy the boundary condition $A_0(\rho = m/2) = 0$. In the case of a massless field, the divergence of $A_\alpha A^\alpha$ at the horizon causes no difficulties as long as the invariant $F_{\alpha\beta}F^{\alpha\beta}$ is finite. This is easily seen from Eq. (6.2) with $\mathsf{m} = 0$. As usual, we take the charge b to be a small parameter in the problem. Then the energy-momentum tensor (6.2) is $O(b^2)$. Hence the same arguments we used in the massless scalar field can be used here to show that the horizon area formula for the Schwarzschild black hole is preserved in linear perturbation theory.

Now, since we take $m/2 \ll \mathsf{m}^{-1}$, the field of the *massive* vector field can be approximated by the solution of the *massless* vector field equation. The massless vector field equation was solved many years ago by Copson [32] who calculated the full electromagnetic 4-potential due to a stationary point charge in the background of a spherical black hole. Making use of this result with the additional requirement that the zeroth component of the vector field vanish on the horizon, we corroborate Linet and Leaute [38] by following the procedure used in [6] and in the previous sections, to calculate the self-energy of the massive vector field with the simple result

$$\frac{1}{2}bA_0^{(b)} = \frac{b^2}{a(1+m/(2a))^4}\frac{m}{2a}. \tag{6.3}$$

The factor $\frac{1}{2}$ takes care of the fact that the object owes part of its energy to its own field. As given earlier by Vilenkin for $m/2 \ll a$ and by Linet and Leaute for all a, this self energy has the same magnitude as in the case of the electromagnetic field, but opposite sign. Electric charges are *repelled* from neutral black holes, while the charges of massive vector fields are *attracted* to them. The implication of this for the issue of entropy bounds is of great importance. A straightforward calculation shows that for constant $\rho > m/2$ and $a \to m/2$, $A_0 \to 0$, namely, as the charge is assimilated, the massive vector field outside the black hole vanishes! This, of course, harmonize with the No-Hair theorem. Since the particle-vector field interaction action is identical to the interaction action given in Eq. (5.14) and based on the result (6.3), the minimal assimilation energy for particles coupled to massive vector fields corresponds to the equal sign in Eq. (5.16) with the replacement $e^2 \to -b^2$ and $q \to 0$. Correct to $O(b^2)$, the minimal horizon area growth is given by Eq. (5.12). Substitution of $\mathcal{E}$ and dividing by $4\hbar$ gives for the entropy of the object.

$$S \leq 2\pi\frac{ER - b^2/2}{\hbar} \tag{6.4}$$

which is precisely the entropy bound (1.2) with $e^2 \to b^2$ and $s = 0$. Therefore, in a sense, the entropy bound (1.2) was generalized to include vector-meson charge b in the same way that it was generalized to include magnetic monopole charge g, $e^2 \to e^2 + g^2 + b^2$. This generalization, however, does not pose any difficulty from a black-hole entropy point of view, since black holes don't posses this quantum number.

The mass of the vector-meson field, m, play an important role in the validity of the refined entropy bound (6.4). As indicated before, the bound (6.4) is correct to $O(\mathsf{m}^2 m^2)$. So if for example we consider the ρ vector-meson ($\mathsf{m} = 770\,MeV$), then the mass of the black hole must be smaller than $m_{Pl}^2/\mathsf{m} \simeq 10^{15}\,gr$, the mass range for mini black holes. However, if m is large then the mass term in Proca's equations (6.1) cannot be neglected. Nevertheless, the field generated by b is now a short range field. Although there is now a contribution to the energy-momentum

tensor from this field, it is localized around the object, and thus can be lumped into its usual energy-momentum tensor. No novel perturbation to the metric arises from this. Hence, b cannot directly perturb the horizon area formula (2.1), and so m is unaffected by slow lowering of the object. Furthermore, no novel potential term is contributed to $\mathcal{E}$ by the field unless the particle is already next to the horizon; otherwise the short range field does not reach down to the horizon and cannot polarize it. Hence the change in horizon area turns out to be b–independent, and b cannot appear in a generic entropy bound.

We conclude that the conjecture, that the entropy bound

$$S \leq 2\pi \frac{\sqrt{E^2R^2 - s^2} - e^2/2}{\hbar} \tag{6.5}$$

for an object with spin s, charge e, maximal radius R and mass-energy $E = \mu$ is the tightest *generic* bound on entropy, seems reasonable.

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APPENDIX A: FIELD OF SCALAR CHARGE IN BLACK HOLE BACKGROUND

Here we determine Φ resulting from a scalar charge b in the Schwarzschild background Eq. (4.3). Using the conventions of Misner, Thorne and Wheeler [14] we write the Klien–Gordon equation for the axisymmetric stationary massless scalar field of a test point scalar charge b situated at $\{\rho, \theta\} = \{a, 0\}$ as

$$(1 - (m/2\rho)^2)\Delta\Phi + (m^2/2\rho^2)\overrightarrow{\rho} \cdot \overrightarrow{\nabla}\Phi = -4\pi b\delta(\rho - a)\delta(\theta)\delta(\phi). \tag{A1}$$

Here Δ and $\overrightarrow{\nabla}$ are the usual Laplacian and Gradient operators in flat spacetime respectively. The potential Φ of the scalar charge b may be looked for in the form [39]

$$\Gamma_a^{-1/2} \left[U_0 + U_1 \Gamma_a + U_2 \Gamma_a^2 + \ldots \right]. \tag{A2}$$

Γ_a here denotes the square of the geodesic distance from the source location in the space whose metric is (4.3), namely $\Gamma_a \equiv \rho^2 + a^2 - 2\rho\, a \cos\theta$. U_0, U_1, $U_2, \ldots$ are analytic functions of ρ for $\rho > m/2$. Let us scale ρ by the rule $\rho \to 2\rho/m$. Substituting the elementary solution (A2) in Eq. (A1) and analyzing the first three terms suggests that instead of determining successively the remaining U_n, we should look for the form

$$\Phi = B\frac{\rho}{\rho^2 - 1}F(\gamma), \qquad \gamma \equiv \Gamma_a/(\rho^2 - 1) \tag{A3}$$

where B is a constant to be determind later. Doing so, we observe that $F(\gamma)$ obeys

$$2\gamma(\gamma + a^2 - 1)\frac{d^2}{d\gamma^2}F(\gamma) + 3(2\gamma + a^2 - 1)\frac{d}{d\gamma}F(\gamma) + 2F(\gamma) = 0. \tag{A4}$$

Therefore the solution of Eq. (A4) is a linear combination of [40]

$$\begin{aligned} F_1 &= \frac{1}{\sqrt{\gamma}\sqrt{\gamma + a^2 - 1}}, \\ F_2 &= \frac{1}{\sqrt{\gamma}\sqrt{\gamma + a^2 - 1}} \log\left(\frac{\sqrt{\gamma} + \sqrt{\gamma + a^2 - 1}}{\sqrt{a^2 - 1}}\right). \end{aligned} \tag{A5}$$

Substituting for γ and F in the definition (A3) and rescaling $\rho \leftarrow 2\rho/m$, we find for Φ

$$\begin{aligned} \Phi_1 &= B_1\frac{\rho}{\sqrt{\Gamma_a \Gamma_{\tilde{a}}}}, \\ \Phi_2 &= B_2\frac{\rho}{\sqrt{\Gamma_a \Gamma_{\tilde{a}}}} \log\left(\frac{\sqrt{\Gamma_a} + \sqrt{\Gamma_{\tilde{a}}}}{\sqrt{1 - (m/2\rho)^2}(2\rho/m)(2a/m)\sqrt{1 - (m/2a)^2}}\right) \end{aligned} \tag{A6}$$

where $\tilde{a} \equiv m^2/4a$.

Our approach is perturbative in nature. Physical invariants that may be assembled from the energy-momentum of the perturbation must thus be bounded everywhere, including at the horizon: any divergence would imply divergence of the curvature invariants. As can be easily verified, every invariant of the geometry associated with this solution is proportional to $(\Phi_{,\alpha}\Phi^{,\alpha})^k$. Now, since $\Phi_{,0}$ is assumed to be identically zero and since we are using the metric (4.3), the mentioned invariant would be bounded provided the solution and its gradient are bounded everywhere. Now, both solutions Φ_1 and Φ_2 vanish at spatial infinity; for $m/2 < a < \infty$ and $\rho \to \infty$ they are $O(1/\rho)$. Furthermore, both solutions are singular at the charge location, $\{\rho, \theta\} = \{a, 0\}$ as required. However, for $a > m/2$, Φ_2 diverges logarithmally everywhere on the horizon $\{\rho = m/2, \forall\theta\}$. We thus reject it as a physical solution.

The remaining solution Φ_1 has some intriguing characteristics. Firstly, for constant $\rho > m/2$ and $a \to m/2$, $\Phi \to 0$, namely, as the charge is assimilated, the scalar field outside the black hole vanishes! (see Eq. (A7) below). This, of course, agrees with the No-Hair theorem for black holes. Secondly, for constant $a > m/2$ and $\rho \to m/2$ the value of the scalar field is finite. It is true that in the limit $a \to m/2$, $\rho \to m/2$ and $\theta \to 0$ Φ_1 diverges. But, one should not be alarmed by this, since this divergence is localized at the point where the scalar charge touches the horizon, $\{\rho, \theta\} = \{a = m/2, 0\}$ and does not encompass the whole of the horizon. Furthermore, this divergence can be attributed to our neglect of the self-energy of the particle.

We thus infer that the elementary solution of Eq. (A1) with source at $\{\rho, \theta\} = \{a, 0\}$ is

$$\Phi(\rho, \theta) = -b\,\frac{1-(m/2a)}{1+(m/2a)}\,\frac{\rho}{\sqrt{\Gamma_a\,\Gamma_{\tilde{a}}}} \tag{A7}$$

in which the constant B_1 was set by the asymptotic value of the field at the position of the charge.

The expression for the scalar field $\Phi(\rho, \theta, \phi)$ due to a charge b situated at the point $\{\rho', \theta', \phi'\}$ can be obtained from Eq. (A7) by a rotation of the axes, which manifests itself by the simple replacement

$$\cos\theta \to \chi(\theta, \phi; \theta'\phi') = \cos\theta\,\cos\theta' + \sin\theta\,\sin\theta'\cos(\phi - \phi'). \tag{A8}$$

The result is analogous to the one in Eq. (A7) with $\Gamma_{\rho'} \to \Gamma_{\rho',\theta',\phi'} \equiv \rho^2 + \rho'^2 - 2\rho\,\rho'\,\chi(\theta, \phi; \theta'\phi')$. This provide the means to calculate the scalar field originating from any arrangement of scalar charges by means of the following formula

$$\Psi\,(\rho, \theta, \phi) = \int\,\Phi(\rho, \theta, \phi; \rho', \theta', \phi')\Sigma(\rho', \theta', \phi')\,\sqrt{-g}\,d^3x' \tag{A9}$$

where $\Sigma(\rho', \theta', \phi')$ is the charge distribution density of a specified scalar charges configuration and the integration is assumed over a constant time slice of the spacetime.

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[1] J. D. Bekenstein, Phys. Rev. D **23**, 287 (1981).
[2] J. D. Bekenstein and M. Schiffer, Int. J. Mod. Phys. C **1**, 355 (1990).
[3] O. Zaslavskii, Class. Quantum Grav. **13**, L7 (1996).
[4] R. D. Sorkin, R. M. Wald and Z. Z. Jiu, Gen. Relativ. Gravit. **13**, 1127 (1981).
[5] O.Zaslavskii, Phys. Lett. **A 160**, 339 (1991).
[6] J. D. Bekenstein and A. E. Mayo Phys. Rev. D (to be published), gr-qc/9903002.
[7] O. Zaslavskii, Gen. Relativ. Gravit. **24**, 973 (1992).
[8] S. Hod, gr-qc/9901035.
[9] J. D. Bekenstein, Phys. Rev. D **30**, 1669 (1984).
[10] J. D. Bekenstein, Phys. Rev. D **7**, 2333 (1973).
[11] J. D. Bekenstein, gr-qc/9808028.
[12] D. Christodoulou, Phys. Rev. Lett. **25**, 1596 (1970).
[13] B. Carter, Phys. Rev. D **174**, 1559 (1968).
[14] C. W. Misner, K. S. Thorne and J. A. Wheeler, *Gravitation* (Freeman, San Francisco, 1973)
[15] R. Hojman and S. Hojman, Phys. Rev. D **15**, 2724 (1977).
[16] S. Hod, gr-qc/9903010; Phys. Rev. D (to be published), gr-qc/9903011.
[17] B. Linet, gr-qc/9903064.

[18] S. Droz, M. Heusler, and N. Straumann, Phys. Lett. **B 268**, 371 (1991); M. Heusler, N. Straumann, and Z. H Zhou, Helv. Phys. Acta **66**, 614 (1993).
[19] M. S. Volkov and D. V. Gal'tsov, JETP Lett. **50**, 312 (1989); **50**, 346 (1990); P. Bizon, Phys. Rev. Lett. **64**, 2844, 1990.
[20] N. Bocharova, K. Bronnikov, and V. Melnikov, Vestn. Mosk. Univ. Fiz., Astron. **6**, 706 (1970); J. D. Bekenstein, Ann. Phys. (NY) **82**, 535 (1974); **91**, 72 (1975).
[21] B. R. Greene, S. D. Mathur, and C. M. O'Neill, Phys. Rev. D **47**, 2242 (1993).
[22] O. Brodbeck and N. Straumann, Phys. Lett. **B 324**, 309 (1996) and J. Math. Phys. **37**, 1414 (1994); N. E. Mavromatos and E. Winstanley, Phys. Rev. D **53**, 3190 (1996); N. Straumann and Z. H. Zhou, Phys. Lett.**B 243**, 33 (1991); Nucl. Phys. **B 369**, 180 (1991); K. A. Bronnikov and Yu. N. Kireyev, Phys. Lett. **A 67**, 95 (1978).
[23] L. Parker, Phys. Rev. D **7**, 976 (1973), and references cited therein.
[24] S. S. Schweber, H. A. Bethe and F. De Hoffman, *Mesons and Fields* (Row Peterson, Evanston, 1955).
[25] J. D. Bekenstein Phys. Rev. D **9**, 3292 (1974).
[26] W. G. Unruh and R. M. Wald, Phys. Rev. D **25**, 942 (1982)
[27] J. D. Bekenstein, Phys. Rev. D **49**, 1912 (1994).
[28] M. A. Pelath and R. M. Wald, Phys. Rev. D (to be published), gr-qc/9901032.
[29] J. D. Bekenstein, gr-qc/9906058.
[30] A. Vilenkin, Phys. Rev. D **20**, 373 (1979).
[31] A. G. Smith and C. M. Will, Phys. Rev. D **22**, 1276 (1980).
[32] E. Copson, Proc. R. Soc. London A**118**, 184 (1928).
[33] B. Linet, C. R. Seances Acad. Sc. A **284**, 215 (1977).
[34] A. I. Zelnikov and V. P. Frolov, Sov. Phys. JETP **55**, 191 (1982)
[35] A. G. Wiseman (unpublished).
[36] L. M. Burko (unpublished).
[37] J. D. Bekenstein, in *Black Holes, Gravitational Radiation and the Universe*, edited by B. R. Iyer and B. Bhawal (Kluwer, Dordrecht 1998); A. E. Mayo, Phys. Rev. D **58**, 104007 (1998).
[38] B. Leaute and B. Linet, Gen. Relativ. Gravit. **17**, 783 (1985).
[39] J. S. Hadamard, *Lectures on Cauchy's Problem in Linear Partial Differential Equations* (Dover, New York, 1952).
[40] This solution can be easily found by the use of any cumputer enviroment for doing mathematics such as Mathematica.